AD-A071 718 FLORIDA UNIV GAINESVILLE DEPT OF INDUSTRIAL AND SYS--ETC F/G 12/2
A MULTI-PRODUCT, 2-STAGE, MULTI-MACHINE SCHEDULING PROBLEM: CON--ETC(U)
APR 79 C S LOVELAND, T J HODGSON N00014-76-C-0096
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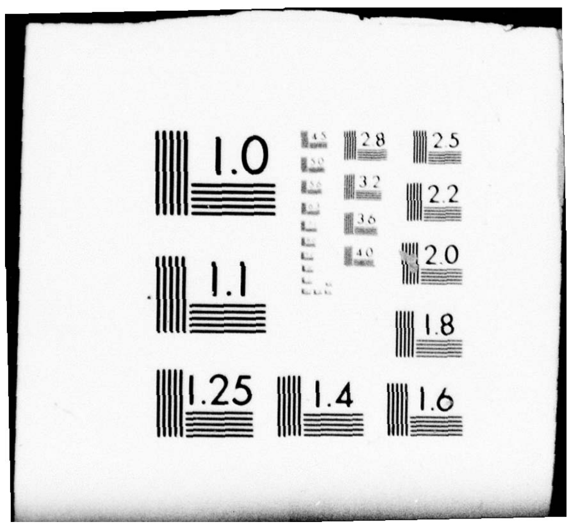
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RESEARCH REPORT

Industrial & Systems
Engineering Department
University of Florida
Gainesville, FL. 32611

(12)

A MULTI-PRODUCT, 2-STAGE, MULTI-MACHINE SCHEDULING PROBLEM: CONDITIONS FOR OPTIMALITY

Research Report No. 79-1

by

C. Stafford Loveland
and
Thom J. Hodgson

April, 1979

Department of Industrial and Systems Engineering
University of Florida
Gainesville, Florida 32611

This research was supported in part by the Office of Naval Research, under contract number N00014-76-C-0096.

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: 79-1
2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER: 9 Research rept.
4. TITLE (and Subtitle): 6 A Multi-Product, 2-Stage, Multi-Machine Scheduling Problem: Conditions for Optimality.
5. TYPE OF REPORT & PERIOD COVERED: Technical
6. PERFORMING ORG. REPORT NUMBER: 79-1
7. AUTHOR(s): 10 C. Stafford Loveland [redacted] Thom J. Hodgson
8. CONTRACT OR GRANT NUMBER(s): 15 N00014-76-C-0096
9. PERFORMING ORGANIZATION NAME AND ADDRESS:
Industrial and Systems Engineering
University of Florida
Gainesville, Florida 32611
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 16 20061102A14D Rsch in & Appl of Applied Math.
11. CONTROLLING OFFICE NAME AND ADDRESS:
Office of Naval Research
Arlington, VA
12. REPORT DATE: 11 Apri[redacted]979
13. NUMBER OF PAGES: 43
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): 12 44p.
15. SECURITY CLASS. (of this report): Unclassified
15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report):
Approved for public release; distribution unlimited.
14 RR-79-1
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report):
N/A
18. SUPPLEMENTARY NOTES
19. KEY WORDS (Continue on reverse side if necessary and identify by block number):
Scheduling
Multi-echelon
Heuristics

404 399 Gu

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper we examine a multi-product, multi-machine, 2-echelon scheduling problem. First, a heuristic approach to solving the problem is developed then a series of parameter conditions are specified which, if met, result in the heuristic providing:

1. A feasible solution if one exists; and
2. an optimal solution if one exists.

Several extensions of the problem are discussed. Finally, solution approaches for other parameter conditions are discussed.

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Abstract

In this paper we examine a multi-product, multi-machine, 2-echelon scheduling problem. First, a heuristic approach to solving the problem is developed then a series of parameter conditions are specified which, if met, result in the heuristic providing:

1. A feasible solution if one exists; and
2. an optimal solution if one exists.

Several extensions of the problem are discussed. Finally, solution approaches for other parameter conditions are discussed.

Keywords: Scheduling, multi-echelon, heuristics.

INTRODUCTION

In this paper, an extension of the multi-facility, multi-product problem examined by Dorsey, Hodgson, and Ratliff [10] is discussed. Where the problem of Dorsey et. al. is concerned with different products manufactured in a single production operation, this problem is concerned with different products manufactured in a series of production operations. Each operation is unique and is called a stage of production. Each product requires the same sequence of production stages in its creation as every other product. Each machine is used exclusively in a given production stage.

For instance, a gear blank for an automobile transmission is first turned and given an initial shape on a screw machine. The gear teeth are cut using a gear hobber. Then the gear goes through a series of machining operations which result in a finished gear, ready for assembly into a transmission. It is also many times the case that more than one product is produced on any given set of facilities (stage). In this situation it is necessary to schedule the products to be produced over the stages in order to satisfy the demand for the products. It is assumed (with little loss in generality) that time can be discretized for scheduling purposes into periods (i.e., shifts, days, weeks). It is also assumed that the demand for each product is known sufficiently well to be used for planning purposes over some horizon. It is this scenario that provides the setting for this paper.

The gear manufacturing example illustrates the requirements of the problem. The different kinds of gears are the different products. Each kind starts from a different kind of blank. The turning of the blank

and the cutting of the gear teeth are different operations and can be considered production stages. Each gear must go through both stages in the same serial sequence. Clearly, the screw machine and the gear hobber can be used only in their respective production stages.

The problem is broken up into two problem types: an anterior and a posterior bottleneck. The anterior bottleneck is characterized by an assumption that any stage of production has <u>less</u> production capacity than any of the succeeding stages. This assumption will cause more time per unit to be devoted to production in the earlier of two stages. The result is that the question of schedule feasibility becomes a question of whether or not there is enough time or machine capacity to handle the demands placed upon the earliest stage of the production system. The posterior bottleneck is characterized by an assumption that any stage of production has <u>more</u> production capacity than any of the succeeding stages.

We first discuss the anterior bottleneck problem with only two stages in series. One possible solution method is presented which consists of solving a series of network flow problems. (A greedy algorithm, developed by Dorsey [10], is presented which solves the network flow subproblems.) It is shown that under certain assumptions the technique finds a feasible solution to the total problem. With the addition of an ordering assumption on the cost coefficients, the technique finds an optimal solution. These results are then extended to N stages in series. Extensions of the model and of the solution technique to more general production systems are briefly discussed. Application of the technique to situations in which demand is uncertain and demand forecasts are used is considered. Finally the posterior bottleneck problem is considered.

Problem Definition

The problem can be described as an industrial process in which M different products are manufactured. Each product undergoes the same two stages of production in the same sequence. Within stage j there are N_j parallel identical machines which perform the operations associated with the stage.

All production runs, called jobs, are performed on a single product on a single machine and have a duration of one time period. An (i,j) job is a production run of stage j of product i. Jobs of the same product and stage may be scheduled consecutively or at the same time on parallel, identical machines. It is assumed either that a setup is included in the production run and is performed for each job or that setups are performed between periods.

Nonnegative in-process and finished-product inventories must be maintained throughout the scheduling horizon H. A newly completed component or finished product is added to the proper inventory at the end of its production period with no time loss for transportation. The necessary raw materials are always available. The components of product i which are used as input for the production of stage 2 of product i during period t are drawn from the stage 1 in-process inventory at the end of period t-1.

Considering the gear manufacturing example, after a batch (job) of type i gear blanks has been turned, the turned blanks are sent to in-process inventory to await input to the gear-hobber stage. Each turned blank becomes one gear in finished inventory after processing by the gear hobber.

The output of stage 2 is the finished product. Demand for each kind of finished product is known for each time period through the scheduling horizon. The demand for period t is satisfied from the finished-product inventory at the end of period t.

The costs of a schedule are incurred in production costs and inventory carrying costs. The production cost for a given stage and product is independent of time. The inventory carrying cost for a given product is a linearly increasing function of time. As a unit of a product finishes a stage of production, its inventory carrying cost per unit increases proportionally with the value added by the stage of production. The objective is to find the production schedule which minimizes the production and inventory carrying costs over the horizon H while satisfying the previously mentioned constraints.

THE LITERATURE

The multi-stage problem has been treated both as a stochastic and as a deterministic problem. A survey of the literature was done by Clark [4] including publications through 1971.

There are five major techniques generally employed on the stochastic problem: expected cost analysis, stationary process analysis, dynamics, dynamic process analysis, and network theory. Berman and Clark [1] and Hadley and Whitin [15, 16] used expected cost models. The same method was used by Gross [14] and Krishman and Rao [22] in considering the problem of Hadley and Whitin. A stationary process analysis appeared in papers by Love [24], Rosenman and Hockstra [28], Sherbrooke [30], and Simon [31]. Clark [3], Clark and Scarf [5, 6], Fukuda [12, 13], Hochstaedter [17], Williams [38], and Zacks [40, 41] applied dynamic programming to stochastic, multi-stage problems. Bessler and Veinott [2] and Ignall and Veinott [18] used Veinott's dynamic process analysis technique. Finally, Connors and Zangwill [7] applied network theory to a stochastic problem.

Deterministic, finite horizon, single-product problems were considered by Crowston and Wagner [8], Kalymon [21], Love [25], Veinott [36], Zangwill [42, 43]. All of their problems had no constraint on the number of machines in a stage. Von Lanzenauer [37], however, treated a deterministic, finite horizon, multi-product problem. The problem has a constraint on the number of machines in each stage and is formulated as a 0-1 program.

Deterministic, infinite horizon problems were considered by Crowston, Wagner, and Williams [9], Jensen and Khan [19], Schwarz and Schrage [29], and Taha and Skeith [34]. Each of these deals with a single product.

Additional contributions on the problem have been made by Evans [11], Johnson and Montgomery [20], Ratliff [27], Sobel [32], Szendrovits [33], Thomas [35], and Young [39].

THE BACKWARD SOLUTION TECHNIQUE

If stage 1 and stage 2 are viewed as two problems, there are two basic differences between them. Stage 2 must have stage 1 supply its input requirements, while stage 1 always has a sufficient amount of raw materials for its input. Stage 2 must satisfy demand with its output, while stage 1 must satisfy the input requirements of stage 2. Assume for the purposes of solving stage 2, that stage 1 is able to meet the input requirements of stage 2. Also, consider the input requirements of stage 2 to be demand on stage 1. If the two stages are viewed in this way and if stage 2 is considered before stage 1, the two stages can be viewed individually as single-stage problems of the type solved by Dorsey et. al. [10]. The ability to perceive the two stages as two single-stage problems suggests the following heuristic procedure, called the Backward Solution Technique or BST, which uses Dorsey's greedy algorithm [10].

Step 1. Set j=2 and use the demand for the stage 2 problem.

Step 2. Solve the problem presented by stage j, using Dorsey's algorithm. Use the result to define the schedule for stage j.

Step 3. If j=2, set j=1. Use the input requirements of stage 2 as the demand for stage 1, and go to Step 2. Otherwise, stop.

An application of the BST to the gear manufacturing problem would cause the teeth-cutting stage to be optimally scheduled first. This schedule would act as a demand timetable for turned gear blanks to be used as input for the gear hobbers. The gear-blank-turning stage is then solved optimally using the demand from the gear hobbers.

It is desirable at this point to present the following list of notation:

b_i^1	inventory carrying cost per batch (job) per period of stage 1 of product i
b_i^2	incremental inventory carrying cost (value added) per batch (job) per period of stage 2 of product i
H	number of periods in the scheduling horizon
I_i^j	desired level for the final inventory of stage j of product i
$I_i^j(0)$	initial inventory level of stage j of product i
M	number of kinds of finished products
N_j	number of identical machines which perform operation j
N(1,2)	minimum number of supplier jobs in stage 1 for each job in stage 2
p_i^j	production rate (batch size) for stage j of product i
$\lfloor a \rfloor$	the largest integer no greater than a

It remains to be shown under what conditions the BST gives an optimal, or even a feasible, solution to the scheduling problem. First, however, Dorsey's algorithm and its use in the BST need to be explained. Dorsey's algorithm is presented primarily because much of the later development in this chapter depends on its structure. (Note that the following discussion considers a single-stage problem.)

In order to present Dorsey's algorithm, the concept of a relative deadline for scheduling a job must be explained first. The relative deadline of a job is that period in the scheduling horizon in which the first unit of the output of the job is used to satisfy demand.

This period can be determined before scheduling takes place. Thus, a job may not be scheduled later than its relative deadline.

In order to determine relative deadlines, it is necessary to be able to distinguish between two jobs of the same product. In some (not necessarily optimal) schedule for a set of identical jobs, number the jobs according to their relative positions in the schedule. If two jobs are in the same period, then the job on the higher-numbered machine has the higher number. Thus, the $n+1^{st}$ job of a product is "later" than the n^{th} job. Without loss of generality the n^{th} job of the product in this schedule is the n^{th} job of the product in any optimal schedule. Also, a first-in-first-out inventory system is assumed for each of the products for simplicity of notation and without loss of generality.

Using numbered jobs and a FIFO inventory system, initial inventory ($I_i^j(0)$) satisfies the early demand. After that the first unit of demand is satisfied by the first unit of output from the first job. The period in which this happens is the relative deadline of the first job, etc. In making these calculations the desired level of the final inventory (I_i^j) is considered part of the demand in the last period.

In his scheduling procedure Dorsey [10] considers the periods one at a time, starting with the last period. Within the period, he starts by scheduling the jobs of the highest-numbered product. He schedules as many as possible of these jobs which have not already been scheduled and which have relative deadlines no earlier than the period under consideration. Having scheduled one product, he starts with the next lower-numbered product. Having completely scheduled a period, he considers the next earlier period.

Consider Table 1A for a two-machine example of the use of Dorsey's method. The entries of the table are the demand (in units) for the two products in the example. Products 1 and 2 have production rates of two and three units per period, respectively. Product 2 has an initial inventory of four units. Since period 4 has a demand for three units of product 1, it will take two jobs to produce them. So the first two product 1 jobs have their relative deadline at period 4. This leaves one unit in product 1 inventory going into period 5. However, that is not enough to cover the demand in period 5 -- another product 1 job is needed and has its relative deadline in period 5. Because of its initial inventory only two jobs are needed to satisfy demand for product 2 in period 4. Table 1B has as its entries the number of jobs of each product which have their relative deadlines in the indicated periods. Table 1B shows that periods 5 through 7 have enough machines so that their jobs can be scheduled at their deadlines. However, period 4 has deadlines for four jobs but room for only two. The scheduling method places the higher-numbered products in period 4 and considers the others in period 3. Figure 1 is a Gantt chart the entries of which are the product numbers of the jobs scheduled in the indicated periods.

Consider again a two-stage problem. In order to use Dorsey's method in a stage other than the last stage of production, it is necessary to be able to use the concept of the relative deadline in an earlier stage. Demand is taken from inventory at the end of the period in which it appears; however, input to a later stage is taken from inventory at the end of the period immediately before it is needed. As a convention to rectify this problem let the input requirement for stage 2 at time t be regarded as

TABLE 1A Single-stage example problem-demand table.
[Units of product]

TIME	1	2	3	4	5	6	7
PRODUCT 1				3	2	1	2
PRODUCT 2				8	4	4	3

TABLE 1B Relative deadlines for the problem in Table 1A.
[Periods of production]

TIME	1	2	3	4	5	6	7
PRODUCT 1				2	1		1
PRODUCT 2				2	1	1	1

TIME	1	2	3	4	5	6	7
STAGE 1	1	1	2	2	1	2	
	1	2	2	2	2	2	
STAGE 2			1	2	1		1
			1	2	2	2	2

FIGURE 1 Two-stage schedule of product 2
[Product numbers of jobs scheduled]

demand on stage 1 at time t-1. The relative deadline for jobs in stage 1 depends on the demand placed on stage 1 by the schedule calculated for stage 2. Dorsey's method can now be applied to stage 1 in a straightforward manner.

Use the example in Tables 1A and 1B and stage 2 of Figure 1. Thus, $p_1^2=2$, $p_2^2=3$, $I_2^2(0)=4$, and $I_1^2=1$. Let stage 1 have two machines, $p_1^1=p_2^1=2$, and $I_2^1=1$. The stage 1 "demand" derived from the stage 2 schedule in Figure 1 is in Table 2A. The demand translates into the stage 1 relative deadlines in Table 2B. From this Dorsey's method derives the stage 1 schedule in Figure 1.

The stage 2 and stage 1 schedules in Figure 1 represent the BST solution to the two-stage problem which uses the demands in Table 1A.

The Backward Solution Technique and the method used to solve the individual stages within the BST were presented in this section. The next section presents the conditions under which the BST finds feasible and optimal solutions to the two-stage problem.

TABLE 2A First stage demand caused by the stage 2 schedule in Figure 1. [Units of product]

TIME	1	2	3	4	5	6	7
PRODUCT 1		4		2		2	
PRODUCT 2			6	3	3	3	

TABLE 2B Relative deadlines for the stage 1 problem in Table 2A. [Periods of production]

TIME	1	2	3	4	5	6	7
PRODUCT 1		2		1		1	
PRODUCT 2			3	2	1	2	

FEASIBILITY AND OPTIMALITY CONDITIONS FOR THE BACKWARD SOLUTION TECHNIQUE

The purpose of this section is to develop conditions which enable the BST to find feasible and (under additional conditions) optimal solutions to the problem of the previous section. Three assumptions will be presented. It will be proven that under the three assumptions, the BST will find a feasible solution, if one exists. A fourth assumption will be made. It will be shown that under the four assumptions the BST will find an optimal solution, if one exists.

The first two $(2,1)^*$ jobs in Figure 1 are supplier jobs of the first $(2,2)^*$ job. A job of stage 1 is a <u>supplier job</u> of a stage 2 job if the first unit of output from the stage 1 job is used as input to the stage 2 job. Also, a job of stage 2 is a <u>consumer job</u> of a stage 1 job if the stage 2 job uses as input the first unit of output from the stage 1 job. Consequently, the first $(2,2)^*$ job in Figure 1 is the consumer job of the first two $(2,1)^*$ jobs, and the second $(2,2)^*$ job is the consumer job of the third $(2,1)^*$ job.

The first assumption to be used in the proof that the BST finds feasible solutions concerns the relative production rates (batch sizes) of the same product between the two stages.

<u>Assumption 1</u>: (production rate) The production rate $p_i^1 \le p_i^2$ for all i.

The consequence of the production rate assumption is that each $(i,2)^*$ job requires an input at least as great as the output of one $(i,1)^*$ job. In fact, each $(i,2)^*$ job requires input equivalent to the output of $p_i^2/p_i^1 (i,1)^*$ jobs.

<u>Assumption 2</u>: (supplier job) Each stage 2 job has at least N(1,2) stage 1 supplier jobs, where $N(1,2) = \min_i \left\lfloor p_i^2/p_i^1 \right\rfloor$.

*(Product#, Stage#)

When Assumption 1 is considered, $N(1,2) \geq 1$, and Assumption 2 states that each stage 2 job has at least one supplier job in stage 1.

In the early periods of the scheduling horizon, demand can cause the first stage 2 job of a product to be scheduled so early that there is not enough time to schedule its supplier jobs. Another possibility is that initial inventory of stage 1 of product i is so high that the first (i,2) job does not need supplier jobs for its input. The supplier job assumption avoids these possibilities by forcing each stage 2 job to have at least N(1,2) supplier jobs. Note that this assumption effectively places an upper bound on initial in-process inventories. Practical considerations of this upper bound will be discussed in a later section. Furthermore, Assumption 2 ensures that if a product is produced in stage 2, it has a component produced in stage 1, since it must have a supplier job.

<u>Assumption 3</u>: (machine availability) The number of stage 1 machines

$$N_1 \leq N_2 N(1,2).$$

Without the truncation in N(1,2), Assumptions 2 and 3 state $N_1 \leq N_2 \min_i (p_i^2/p_i^1)$ or $p_i^1 N_1 \leq p_i^2 N_2$ for all i. Thus, the one-period production capability of stage 1 is no greater than the one-period production capability of stage 2 for any product and the production bottleneck is maintained in stage 1. The consequence of the machine availability assumption, when combined with Assumptions 1 and 2, is that there are no more machines available in any period t of stage 1 than are necessary to execute the minimum number of supplier jobs which N_2 jobs in period t+1 of stage 2 can have. In other words, if there are no idle machines in period t+1 of stage 2 and all stage 1 jobs are scheduled as late as possible, there are no idle machines in period t of stage 1.

It is shown in [26] that violation of any of the Assumptions 1, 2, or 3 can result in the BST failing to find a feasible solution, when one exists. It will be shown that when all three assumptions are met, the BST finds a feasible solution, if one exists. First, however, some lemmas must be proven.

The first lemma characterizes a solution to a single-stage problem obtained by Dorsey's method. It concerns the ordering of the jobs by product number within a schedule.

Lemma 1: Consider a schedule, found by Dorsey's algorithm, to a single-stage problem. If a job of some product i is scheduled in period t1 earlier than its relative deadline t2, then every machine in the interval [t1+1, t2] is utilized by the schedule, and every job in the interval has a product number which is at least as high as i.

Proof: Assume for the purpose of contradiction that a machine in some period t in the interval [t1+1, t2] is idle or scheduled to process a job of product i1, where i1 < i. When Dorsey's algorithm scheduled period t, it assigned to t all product i jobs which were unscheduled and had relative deadlines no earlier than t. This assignment was made before any jobs of product i1 were assigned to t and before any decision was made to leave a machine in t idle. Consequently, no job of product i which has a relative deadline in period t or later is scheduled earlier than period t. This contradicts the existence of the product i job referred to in the hypothesis of the lemma. Therefore, every machine in period t is utilized, and every job in period t has a product number which is at least as high as i, for all t in [t1+1, t2].

Q.E.D.

The insight to be found in Lemma 1 is that in a Gantt chart of Dorsey's solution all the machine blocks between a given job and its relative deadline contain jobs having product numbers no lower than the product number of the job under consideration.

Lemma 2: If there exists a feasible solution to a single-stage problem, then Dorsey's algorithm will find a unique, feasible solution.

Proof: Follows from Dorsey [10].

Q.E.D.

The preceding assumptions and lemmas are used in Lemma 3, which shows that, under certain conditions, a pairwise interchange of two jobs in the second stage of a feasible schedule for a two-stage problem can be made so that the resulting schedule is feasible.

In considering Lemma 3, refer to the Gantt chart in Figure 2. Unlike the previous Gantt chart, each stage has an unspecified number of machines, except that they satisfy the machine availability assumption. The i3 in some of the periods represents a class of products (not necessarily all of which have the same product number) all of which have higher product numbers than i1 and i2. The i4 in other periods represents another class of products. The i1 and i2 in the chart indicate that each of those periods contains at least one job of product i1 or i2, among other jobs.

Note in the proof of the lemma that all changes in the solution are accomplished by applying Dorsey's algorithm to a stage or by a pairwise interchange of jobs in which the job of the higher-numbered product moves to a later period and the job of the lower-numbered product moves to an earlier period.

Lemma 3: Consider the Gantt chart of a feasible solution to a two-stage problem in which the first stage has a feasible solution

TIME	t1	...	t2	...	t3-1	t3	...	t4-1	t4	...	t5
STAGE 1	i2	...	i4	...	i4	i4	...	i1		...	
STAGE 2		...	i2	...	i3	i3	...	i3	i1	...	

FIGURE 2 Portions of a solution for two adjacent stages [Product numbers of jobs scheduled]

found by Dorsey's algorithm and in which each stage 2 job is scheduled so that there are no idle machines between it and its relative deadline. For any two products i1 and i2 such that i1 < i2, consider an (i1, 2) job in some period t4 and an (i2, 2) job in some period t2 (t2 < t4) both of which have relative deadlines no earlier than t4. Furthermore, let i1 be the lowest product number in period t4 of stage 2 and let this (i1, 2) job be the "earliest" (lowest job number) of the (i1, 2) jobs in period t4. Finally, let the (i2, 2) job be the "latest" (highest job number) of the (i2, 2) jobs in period t2. If

(a) the jobs in the interval [t2+1, t4-1] of stage 2 are of the product class i3,

(b) the (i1, 2) job and the (i2, 2) job are interchanged in the schedule, and

(c) Assumptions 1, 2, and 3 are met,

then there exists a feasible solution which is identical to the new solution in stage 2 and has a feasible solution from Dorsey's algorithm in stage 1.

<u>Proof</u>: It will be shown that all the (i1, 1) supplier jobs of the (i1, 2) job which are in the interval [t2, t4-1] are actually in period t4-1. It will also be shown that there are no more than N(1, 2) of these (i1, 1) supplier jobs in period t4-1. Thus, when the interchange of their (i1, 2) consumer job and the (i2, 2) job is made, the N(1, 2) (i1, 1) supplier jobs in t4-1 can interchange feasibly with N(1, 2) (i2, 1) supplier jobs of the (i2, 2) job in periods earlier than t2. The resulting solution is feasible. Finally, Dorsey's algorithm is applied to the new stage 1 solution.

Consider Figure 2. Let period t1(<t2) be the latest period containing (i2, 1) supplier jobs of the (i2, 2) job in period t2. Define all the stage 1 jobs in the interval [t1+1, t4-2] to be in the product class i4. The (i2, 1) supplier job in period t1 has its relative deadline in period t2-1. By Lemma 1, each job in the interval [t1+1, t2-1] has a product number no lower than i2, therefore, higher than i1. By Assumptions 1 and 2, each stage 2 job in the interval [t2+1, t4-1] has at least N(1, 2) supplier jobs. Then, by Assumption 3 and Lemma 1, all stage 1 jobs in the interval [t2, t4-2] have product numbers at least as high as those of the products in class i3, which are higher than i1 and i2. Therefore, all products in the class i4 have numbers at least as high as i2 and higher than i1. Consequently, there are no (i1, 1) jobs in the interval [t1+1, t4-2].

Since the (i1, 2) job of interest in period t4 is the lowest-numbered job of the lowest-numbered product in period t4 and, by hypothesis, there are no idle machines between the (i2, 2) job in period t2 and its relative deadline, then period t4 of stage 2 has N_2 jobs, N_2-1 of which have higher product numbers or higher job numbers than the (i1, 2) job under consideration. By all three assumptions and Lemma 1, there are at least (N_2-1)N(1, 2) supplier jobs which have higher product or job numbers than and are scheduled later than the supplier jobs of the (i1, 2) job (N(1, 2) for each of the other jobs in period t4 of stage 2). Therefore, period t4-1 contains <u>at most</u> N(1, 2) supplier jobs of the "earliest" (lowest-numbered) (i1, 2) job in period t4.

Interchange the highest-numbered (i2, 2) job in period t2 and the lowest-numbered (i1, 2) job in period t4. (The interchange is feasible, since each job has its relative deadline at least as late as period t4.)

It has been shown that the (i1, 2) job in the interchange has at most N(1, 2) supplier jobs in the interval [t1+1, t4-1] (all in period t4-1). By the supplier job assumption, the (i2, 2) job in the interchange has at least N(1, 2) supplier jobs in period t1 or earlier. In order to maintain feasibility in stage 1 in conjunction with the stage 2 interchange, the period t4-1 supplier jobs of the (i1, 2) job in the stage 2 interchange must be interchanged with the "latest" supplier jobs of the (i2, 2) job in the stage 2 interchange.

These stage 1 interchanges must preserve the ordering of the jobs of the same product. There are two cases to be considered.

Case 1: There are no (i2, 1) jobs in the interval [t1, t4-2] which are "later" (higher-numbered) than the supplier jobs of the (i2, 2) job in the interchange.

Case 2: There are (i2, 1) jobs in the interval [t1, t4-2] which are "later" (higher-numbered) than the supplier jobs of the (i2, 2) job in the interchange.

If Case 1 is true, moving the (i2, 1) supplier jobs to period t4-1 does not alter the ordering of the (i2, 1) jobs. By Lemma 1, no (i1, 1) jobs lie between the supplier jobs of the (i2, 2) job in the stage 2 interchange and the supplier job's former relative deadline, period t2. Consequently, moving the (i1, 1) supplier jobs to the locations of the (i2, 1) supplier jobs does not alter the ordering of the (i1, 1) jobs. Therefore, make the necessary stage 1 pairwise interchanges.

If Case 2 is true, it is shown in Lemma 1 that there are no (i1, 1) jobs in period t4-1. No stage 1 interchanges are necessary.

After the stage 2 interchange and the stage 1 interchanges (if any are necessary) are made, both stages 1 and 2 have feasible schedules.

The only thing which remains to be done is to convert the stage 1 feasible solution to a feasible solution based on Dorsey's algorithm. By Lemma 2, applying Dorsey's algorithm to the stage 1 problem created by the new stage 2 solution completes the proof.

Q.E.D.

The results of Lemma 3 form the cornerstone of the proof of the following theorem:

Theorem 1: If there exists a feasible solution to the two-stage problem and if Assumptions 1, 2, and 3 are met, the Backward Solution Technique finds a feasible solution.

Proof: The proof consists of assuming the existence of a feasible solution to a two-stage problem and then converting it to a feasible solution both stages of which are Dorsey solutions. After transforming stage 1 into a Dorsey solution, the stage 2 feasible solution is compared to a Dorsey solution of stage 2. Starting with the last period, the stage 2 solution is converted by pairwise interchanges to the Dorsey solution, period by period. After each interchange the stage 1 Dorsey solution is updated to maintain feasibility.

Consider any feasible solution to the two-stage problem. By Lemma 2, stage 1 of the solution can be converted to a Dorsey solution. The solution now is feasible in stage 2 and has a Dorsey solution in stage 1. Order the jobs on the machines in each period of stage 2 by increasing product and job number. If there are any idle machines in stage 2 between a job and its relative deadline, move the job to the latest such machine.

Also consider the Dorsey solution to stage 2. Compare the stage 2 feasible and Dorsey solutions. (The conditions of Figure 2 and Lemma 3 will now be created.) Find the latest period in which the two stage 2 Gantt charts don't agree. Call that period t5. In period t5 of the stage 2 Dorsey schedule find the highest-numbered job of the highest-numbered product which is not in period t5 of the stage 2 feasible solution. Call that job n2 of product i2. Find the period in the stage 2 feasible solution which contains job n2 of product i2. Call that period t2. Since the two stage 2 schedules agree in the interval [t5+1, H], t2 < t5. By the ordering of job numbers and the method of choice of job n2, job n2+1 of product i2 is in the interval [t5, H] in both stage 2 schedules. Thus, job n2 of product i2 is the highest-numbered (i2, 2) job in period t2 of the feasible schedule. Job n2 of product i2 must now work its way, through a series of pairwise interchanges, to period t5 of stage 2 of the feasible solution. (Until it reaches t5, all discussion concerns the feasible solution only.)

The next step is to find a job with which to interchange job n2 of product i2. Find the earliest period which is later than t2 and which contains a product number smaller than i2. Call the period t4 and call the product number i1 (t2 < t4 and i1 < i2). Let job n1 be the lowest-numbered (i1, 2) job in period t4. Let i3 be the class of products in the stage 2 interval [t2+1, t4-1]. By the choice of t4 and job number ordering, all the products in the class i3 have higher numbers than i2. Interchange job n2 of product i2 and job n1 of product i1. Find the new feasible Dorsey schedule for stage 1, the existence of which is shown in Lemma 3.

If $t4 < t5$, job n2 is the only (i2, 2) job in period t4 (job n2+1 is in period t5 or later). Redefine t2 to be the old t4. Search for a new period t4, product i1, and job n1.

When job n2 reaches period t5 ($t4 = t5$), find a new period t5, product i2, and job n2. The search must eventually end with the conversion of period 1 into period 1 of the stage 2 Dorsey solution, because the interval [t5, H] has no schedule alterations (there is no cycling in the search).

The feasible solution now is a Dorsey solution in both stages. Lemma 2 shows that Dorsey's algorithm finds unique solutions. Since the BST finds Dorsey solutions at each stage, this two-stage solution is the one the BST would find.

Q.E.D.

It remains to be shown under what additional conditions the BST will find an optimal solution. The fourth assumption requires that the products can be ordered equivalently in each stage by their cost b_i^j.

<u>Assumption 4</u>: (cost) The inventory cost function $b_i^j \leq b_{i+1}^j$ for

$$i = 1, \ldots, M-1 \text{ and } j = 1, 2.$$

The meaning of the cost assumption in the first stage is that the inventory carrying cost of a batch of product i is no greater than the inventory carrying cost of a batch of product i+1. In stage 2 the cost is the value added to a batch, but the relationship between products must still hold.

The assumption of the relationship between costs, when considered for an individual stage, is the same assumption Dorsey [10] makes to ensure that his method finds an optimal solution. Thus, under the cost assumption,

the BST finds a solution which is optimal in each stage, if a solution exists for each stage. However, that does not mean the total solution is optimal.

Theorem 2: If the Assumptions 1, 2, 3, and 4 are met and if a feasible solution exists to the two-stage problem, then the Backward Solution Technique finds an optimal solution.

Proof: Consider any optimal solution. Clearly, since it is optimal, no idle machines lie between a job and its relative deadline. Convert the optimal solution to a solution which has Dorsey solutions in each stage by the same conversion method as is used in the proof of Theorem 1. There are four ways in which the solution is changed. Reordering the jobs within a stage does not change their objective value. Since there are no idle machines between a job and its relative deadline, the jobs of stage 2 will not be moved later to fill idle machines. In the proofs of Lemma 3 and Theorem 1 all pairwise interchanges moved a job of a higher-numbered product (i2) later and a job of a lower-numbered product (i1) earlier, for a net objective value change per period moved of $b^j_{i2} - b^j_{i1} \geq 0$ (by Assumption 4). Finally, stage 1 is periodically solved using Dorsey's algorithm. Dorsey [10] showed that, under Assumption 4 applied to a single stage, his algorithm finds an optimal solution. Thus, under Assumption 4, applying Dorsey's method to a solution in stage 1 does not worsen the solution.

It has been shown that each alteration which must be made to an optimal solution to a two-stage problem in order to convert it to a solution having Dorsey solutions in each stage results in a solution at least as good as the optimal solution. Consequently, under Assumptions 1, 2, 3, and 4, a solution which has a Dorsey solution in both of its stages is optimal (if a feasible solution exists). Since the BST generates unique

Dorsey solutions in each stage and finds a feasible solution (Lemma 2 and Theorem 1), it finds that optimal solution.

Q.E.D.

It has been shown that under the production rate, supplier job, and machine availability assumptions the BST finds a feasible solution to the two-stage problem (if a feasible solution exists), regardless of objective function. With the addition of the cost assumption, the BST finds the optimal solution. In the next section, practical applications of the BST are considered, such as problems in which demand may fluctuate and/or initial inventory levels may cause the violation of the supplier job assumption.

APPLICATION OF THE BACKWARD SOLUTION TECHNIQUE

The normal use of a scheduling technique like the BST is to schedule production over a horizon using all available information on demand and inventory. After performing one period's production, the schedule is then recomputed using updated demand and inventory information. This cycle is repeated for the remainder of the production process.

In computing the schedules it becomes apparent that initial, in-process inventory may cause Assumption 2, that each (i, j) job has at least N(j, j+1)(i, j) supplier jobs, not to be realized for early jobs. In a practical sense, this situation is a start-up problem, however, and after a period of time resolves itself. Let H_1 be the period in the first schedule by which each product has had completed at least one stage 2 job having a supplier job. From no later than H_1 on, Assumption 2 is satisfied in the first schedule, which must be optimal for all stage 2 jobs scheduled after H_1 and for their suppliers. Thus, the start-up period ends no later than period H_1.

When demand remains unchanged, it is easily seen that any schedule computed after period H_1 is optimal. This scheduling technique loses its effectiveness when large fluctuations in demand are encountered from schedule to schedule. Under the assumption of reasonable accuracy in the forecast of demand, it will be shown that the Backward Solution Technique creates optimal schedules.

When computing a new schedule at time t, compute it from time 0. Then use the portion from t to H. The inventory at time t is the new initial inventory. The schedule from H_1 to H satisfies Assumption 2 and is optimal. Thus, the schedule from t to H is optimal. Let t_i be the earliest period in this schedule in which product i has had completed at

least one stage 2 job having a supplier job. The half-open interval $[t, t_i]$ is the <u>interval of accuracy</u> for the demand forecast for product i.

An increase in demand has two possible impacts on the schedule. There may be enough idle time in the necessary places to handle the added load; thus, Assumption 2 is maintained. Since the addition of a new job to a BST schedule can cause the jobs of lower-numbered products to be pushed to earlier periods, the alternative impact of an increase in demand is the creation of an infeasible problem. If an increase in demand is for a period earlier than H_1 and is feasible, it may cause a redefinition of H_1 to an earlier period. The same is true for period t_i.

A decrease in demand of one or less stage 2 jobs between schedules also has two possible impacts on the schedule. If the decrease in demand for product i applies to a period no earlier than t_i, it is easily shown that the affected jobs in the interval $[t, t_i)$ will maintain the same relative ordering as they advance in the schedule, as a result of the removal of the product i jobs. Thus, the schedule remains optimal. If the decrease in demand for product i occurs for a period earlier than t_i, the schedule may revert to a start-up mode.

In order for the scheduling method to maintain optimality after the start-up period, all decreases in demand for product i must occur no earlier than t_i and must be able to be satisfied by the cancellation of jobs which have a supplier job. This is accomplished when the demand forecast for product i is a lower bound on actual demand in the interval.

Conditions have been developed under which the BST gives an optimal solution. It was shown using worst case analysis that after an initial start-up period the method can accommodate normal changes in demand. A discussion of the extension of the production system to more

general configurations of stages and the application of the BST to the extended system is discussed next.

EXTENSIONS OF THE PRODUCTION SYSTEM

A natural extension of the two-stage problem is to N stages in series. There is no intuitive difference between the two problems. All facets of the definition and solution of the new problem are the same, except that there are N stages to consider. The range of the stage indicator j is now from 1 to N. Any relationships between stages 1 and 2 also apply between stages j and j+1. The production rate, supplier job, machine availability, and cost assumptions are extended in this manner. The proofs required to show that the BST finds both a feasible solution (if one exists) under the first three assumptions and an optimal solution under all four assumptions are inductions on the proofs for the two-stage problem. The appropriate assumptions, theorems, and proofs for the N-stages-in-series problem can be found in [26].

A further extension of the production systems to which the BST can be successfully applied is also discussed in [26]. The new system can be represented by a general, directed, acyclic network of production stage. With the exception of a tightening of the machine availability assumption, all the changes necessary to accommodate the more complex system are concerned with notation.

A POSTERIOR BOTTLENECK PROBLEM

In the earlier discussion of the two-stage problem, Assumption 1 ($p_i^1 \geq p_i^2$) caused each stage 2 job to require input equivalent to the output of at least one stage 1 job. It is shown by counterexample in [26] that the BST may not solve the problem when Assumption 1 is violated. In the following, Assumptions 1 and 3 are modified. The result is that the two-stage problem can be condensed to a single stage problem.

Assumption 1A: $p_i^1 \geq p_i^2$ for all i.

This assumption causes each stage 2 job to require input equivalent to the output of no more than one stage 1 job. Consequently, each stage 2 job has no more than one supplier job. In fact, a stage 1 job may supply more than one stage 2 job. Thus, some jobs may have no supplier jobs. In addition, let us place a lower bound on the number of machines in stage 1 (an alteration of Assumption 3).

Assumption 3A: $N_1 \geq N_2$.

Assumptions 1A and 3A ensure that each supplier job is scheduled at its relative deadline in the optimal solution, as long as the objective function (which is to be maximized) for each job is an increasing function of time. Thus, the bottleneck is in stage 2 (posterior bottleneck). Consequently, the position of the consumer job in the optimal solution completely determines that of its supplier. Finding the stage 2 portion of the optimal solution is equivalent to finding the full optimal solution.

The ordering, as applied in the BST section earlier, of the jobs of a product for each stage can be considered for stage 2. The ordering does not change the optimal schedule. It merely numbers the jobs ahead of time and ensures that they are in order in all, including the optimal, schedules. The effect of the ordering for stage 2 is to designate a

first (i, j) job, a second (i, j) job, etc. A precedence constraint can be used to ensure that the first (i, j) job can be scheduled no later than the second (i, j) job, etc. An example of this appears in Figure 3A.

Figure 3A shows a two-stage production system having one machine per stage. There is a single product which uses both stages. Figure 3A is a Gantt chart in which the numbers are job numbers for that stage of the single product. The production rates, as shown in Figure 3A, are such that job 1 in stage 1 supplies the input to both jobs 1 and 2 in stage 2. Thus, it must be scheduled earlier than the earlier of those two jobs in stage 2. A precedence constraint would require job 1 to be performed no later than job 2 in stage 2.

If the above ordering is kept, a job in stage 1 is in period t if and only if its consumer job is in period t+1. Thus, moving a job in stage 2 causes all of its supplier jobs to move with it by the same number of periods. It is reasonable, therefore, to add the cost of the stage 1 job to that of its consumer job and remove the stage 1 job from the problem. If this is done for all stage 1 jobs, this two-stage problem becomes a single-stage problem which has precedence relationships among the jobs of the same product. Figure 3B demonstrates the reduction of the problem of Figure 3A from two stages to a single stage problem having cumulative costs and a precedence constraint between jobs 1 and 2.

In the new single-stage problem, there is a precedence constraint among the jobs (due to the within-product ordering), and each job has a due-date constraint determined by the demand. The objective function for each job is an increasing function of time.

It may appear that this is a relatively easy problem to solve. However, Lenstra [23] has shown that this problem is NP-complete. Loveland [26] discusses solution techniques for this problem.

TIME	1	2	3	4
STAGE 1		$\boxed{1}$		
STAGE 2			$\boxed{1}$	$\boxed{2}$

$p_1^1 = 2,\ b_1^1 = 20$

$p_1^2 = 1,\ b_1^2 = 30$

FIGURE 3A Solution for single product, 2 stages.

TIME	1	2	3	4
STAGE 2			$\boxed{1}$	$\boxed{2}$

$p_1^3 = 1,\ b_{1,1}^3 = 50,\ b_{1,2}^3 = 30$

FIGURE 3B Solution for Figure 3A reduced to 1 stage.

The extension of the discussion to a production system consisting of N stages in series is straightforward. Assumptions 1A and 3A refer to any two adjacent stages. By repeatedly merging the earliest stage into the next stage, the problem can be condensed to N-1 stages, then N-2, etc. In this way the problem is condensed to a single stage. Finally, the BST can be applied to a more general production system in which the stages may occur in configurations other than series [26].

CONCLUSION

An extension of a single-stage, multi-machine, multi-product scheduling problem into a multi-stage problem was discussed in this paper. The two-stage problem was considered first. A method, called the Backward Solution Technique, was introduced which solves each stage with a greedy algorithm developed by Dorsey [10]. It was found that when the bottleneck occurs in the initial production stage (the production rate and machine availability assumptions) and when production start-up effects are over (the supplier job assumption), the BST finds a feasible solution, if one exists. With the addition of the cost assumption, the BST finds an optimal solution. Practical application of the BST was discussed. It was found that after a start-up period and with reasonably accurate demand forecasts, the production system met the requirements for the effective use of the BST.

After the two-stage problem an extension to N stages in series was discussed. In the final extension, the production system formed a general, directed, acyclic network. With few basic changes [26], the assumptions, lemmas, and theorems remained the same for each of the extensions.

The next logical avenue of exploration was the case of the two-stage problem in which $p_i^1 \geq p_i^2$. This moved the production bottleneck to the last stage. It was shown that with a lower bound on the number of available machines this posterior bottleneck problem can be reformulated as a single-stage problem. Extensions of the two-stage problem can also be reformulated as a single-stage problem. Heuristics to solve the single-stage problem are presented in [26].

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